Discrete Mathematics 38 (1982) 17–21
North-Holland Publishing Company

THE CENTRAL RATIO OF A GRAPH

Fred BUCKLEY

Math. and Science Division, St. John's University, Staten Island, NY 10301, USA

Received 21 February 1979
Revised 2 July 1980 and 26 January 1981

We examine several graph equations involving the center, $C(G)$, of a graph G. The central ratio of G, denoted $c(G)$, is the ratio of $|C(G)|$ to $|V(G)|$. We show that for any rational number r, where $0<r\leq 1$, there is a graph G with $c(G)=r$. For all such r, we describe a corresponding minimal graph. Graphs for which $c(G)=1$ are called self-centered. We give the range of values for $|E(G)|$ for self-centered connected graphs on n vertices. We then characterize all trees whose center vertices get interchanged under complementation.

1. Introduction

In a graph, there are some vertices which are "close" to all of the remaining vertices in the graph; however, certain vertices may be quite "far" from one another. The center of a graph G is defined to be a particular subset of the vertex set, $V(G)$, enjoying the first property. In this paper, we examine several graph equations relating to the center of a graph.

If v and w are in $V(G)$, the *distance between* v and w, $d(v, w)$, is the length of a shortest path from v to w. The *radius* of G is defined as

$$r(G)=\min_v\left(\max_w d(v, w)\right).$$

The *center* $C(G)$ of the graph G is the set of all vertices v satisfying $\max_w d(v, w)=r(G)$. For various other centrality sets, the reader is referred to [5].

For $|V(G)|=n$, we give the range of values for the size of the edge set, $E(G)$, when $C(G)=V(G)$ and G is connected. We show that for any rational number c, where $0<c\leq 1$, there is a graph G for which the ratio of $|C(G)|$ to $|V(G)|$ is c, and we describe a corresponding minimal graph for each such c. Finally, we characterize all trees whose center vertices get interchanged under complementation.

2. The central ratio

We define the *central ratio* $c(G)$ of a graph G to be $|C(G)|\div|V(G)|$. We remark that $0<c(G)\leq 1$. Capobianco has named graphs with central ratio equal to one,

self-centered graphs. Recall that a *block* in G is a maximal connected subgraph without a cutpoint.

Theorem 1 (Harary and Norman). *The center of any connected graph lies in a block.*

Corollary 1. *If G is self-centered, then G is disconnected or G has only one block.*

Theorem 2. *If a/b is a rational number in lowest terms and $0 < a/b \leq 1$, then there exists a graph G with $c(G) = a/b$. Any smallest such graph G^* has b vertices unless $b = a+1$, in which case it has $2b$ vertices and $2b+1$ edges. If $b \neq a+1$, then G^* has b edges, unless $a = 1$ or 2, in which case it has $b-1$ edges.*

Proof. First, let $b = a + k$ where $k > 2$. The graphs of Fig. 1 each have b vertices and b edges. If a is odd and $a > 1$, then (i) is a minimal graph G^*. If a is even and $a > 2$, then the graph (ii) is G^*.

Let $b = a + 2$. Then a is odd, since a/b is in lowest terms. If $a > 1$, then the graph formed by adding a pendant edge to C_{a+1} is a minimal G^*.

Let $b = a + 1$. Some pair of vertices in G are the endpoints of a diameter, and these two vertices are not in $C(G)$. Therefore G must contain at least $2b$ vertices. Such a graph with $2b$ vertices and $2b + 1$ edges is shown in Fig. 2.

Finally, if $b \neq a + 1$ and $a = 1$, then $K_{1,b-1}$ is G^*. If $b \neq a + 1$ and $a = 2$, then the graph formed by adding a pendant edge to one of the endpoints of $K_{1,b-2}$ is a minimal G^*.

Clearly, the graphs described have the least possible number of vertices. To see that these graphs also have the least possible number of edges, we observe that the minimal graphs must be connected. Thus the G^* for $a = 1$ and 2 are minimal. If $a \geq 3$, then it follows from Theorem 1, that G has a cycle, so $|E(G)| \geq |V(G)|$.

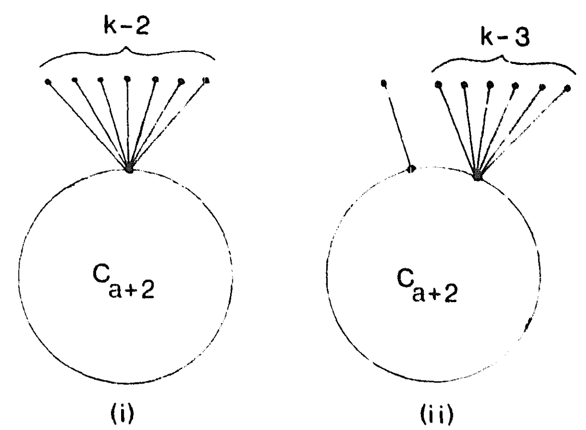

Fig. 1.

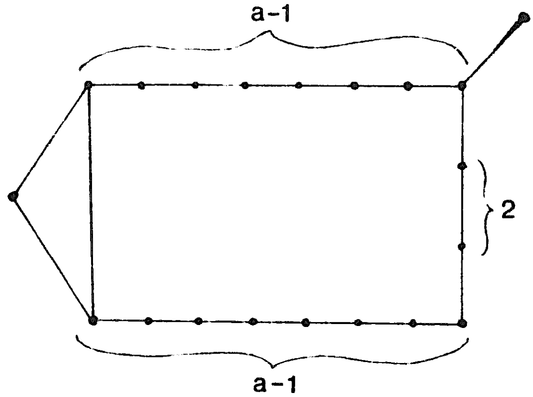

Fig. 2.

Thus the G^* given for $b \neq a+1$ are minimal. If $b = a+1$, then either there are no vertices of degree 1 and no cutpoints, or there is precisely one vertex v of degree 1. In the former case, G must have at least $2b+1$ edges. In the latter case, $G-v$ has an odd number of vertices and can have no cutpoints. Since there is exactly one vertex of $G-v$ which is not in $C(G)$, the graph $G-v$ must have at least $|V(G-v)|+1$ edges. Thus G must have at least $2b+1$ edges. Thus if $b=a+1$, then $|E(G)|=2b+1$, and the G^* shown in Fig. 2 is minimal.

As we mentioned, a graph with central ratio equal to one is called self-centered. In our next theorem, we describe the range of values for the size of the edge set for a connected self-centered graph. Recall: $\lfloor x \rfloor$ is the greatest integer less than or equal to x. The least integer greater than or equal to x is denoted $\lceil x \rceil$.

Definition. If $r \geq 4$, the graph $C_r * sP_2$ is formed from C_r by joining two vertices of C_r at distance two from one another by s additional paths of length two.

Theorem 3. *There exists a connected self-centered graph on n vertices and k edges if and only if $k = \binom{n}{2}$ or $n \leq k \leq \lfloor \frac{1}{2}n(n-2) \rfloor$.*

Proof. We first show that the given values are the only allowable ones. Then we construct a connected self-centered graph for each k described. For a tree T, it is well-known that $\langle C(T) \rangle \cong K_1$ or K_2. Thus if T is a self-centered tree, then $k = \binom{n}{2}$. Let G be connected, have a vertex of degree $n-1$, and $G \neq K_n$. Then $C(G)$ consists precisely of the vertices of degree $n-1$ and G cannot be self-centered. Therefore G is not self-centered if

$$\binom{n}{2} - \left\lceil \frac{n}{2} \right\rceil < k < \binom{n}{2}.$$

Note that

$$\binom{n}{2} - \lceil \tfrac{1}{2}n \rceil = \lfloor \tfrac{1}{2}n(n-2) \rfloor.$$

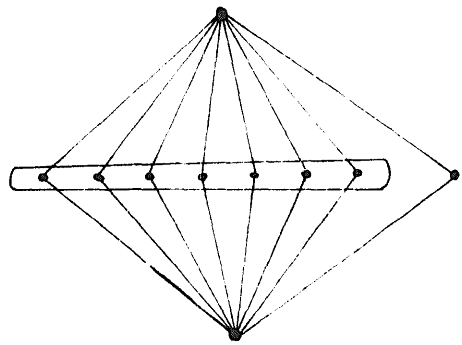

Fig. 3.

The graphs $C_n, C_{n-1} * P_2, C_{n-2} * 2P_2, \ldots, C_4 * (n-4)P_2$ are self-centered on n vertices and have $n, n+1, \ldots, 2n-4$ edges, respectively. From $C_4 * (n-4)P_2$, one may construct self-centered graphs for $2n-5 \leq k \leq \frac{1}{2}(n^2-3n+4)$ by successively joining nonadjacent vertices in the circled region of Fig. 3.

The graphs for $\frac{1}{2}(n^2-3n+4) \leq k \leq \lfloor\frac{1}{2}n(n-2)\rfloor$ may be constructed by concentrating on a spanning cycle S of K_n. We delete a maximum matching from S, and if n is odd, we delete an additional edge. The resulting graph is self-centered and has $\lfloor\frac{1}{2}n(n-2)\rfloor$ edges. We construct all remaining required graphs by successively deleting from K_n edges remaining along S.

Using results of [4] and [6], Theorem 3 was extended by Buckley [1] to the situation where the radius of the graph is known.

3. A graph equation for centers of trees

Our interest in the central ratio was aroused by considering graph equations [2]. Indeed, the central ratio is defined by a graph equation. If f and g are graph functions and H is a class of graphs, then the solution to the graph equation $f(H) = g(H)$ is the set of all graphs in the class H which satisfy the equation.

For the class of trees, T, the only solutions to the equation $C(T) = V(T)$ are K_1 and K_2. The next result characterizes trees solving the equation $C(\bar{T}) = V(T) - C(T)$.

Theorem 4. *A tree T satisfies the equation $C(\bar{T}) = V(T) - C(T)$ if and only if diameter$(T) = 3$.*

Finding a useful characterization of self-centered graphs is a difficult task. The reader looking for an interesting problem might consider this. A more tractable problem is to try to characterize these graphs for various classes. In particular, one might try to characterize these in the case where some graph parameter, for example, connectivity, is known. These cases correspond to solving graph equations simultaneously.

Acknowledgement

The author is grateful to Michael Capobianco for introducing him to graph equations and self-centered graphs.

References

[1] F. Buckley, Self-centered graphs with a given radius, in: F. Hoffman, ed., Proc. 10th S.-E. Conf. Combinatorics, Graph Theory, and Computing (Utilitas Mathematica, Winnipeg, 1979) 211–215.
[2] M. Capobianco, Graph Equations, in: A. Gewirtz and L. V. Quintas, eds., Proc. 2nd Internat. Conf. in Combinatorics (N.Y. Acad. Sci., New York, 1979) 114–118.
[3] F. Harary and R.Z. Norman, The dissimilarity characteristic of Husimi trees, Ann. Math. 58 (1953) 134–141.
[4] P.A. Ostrand, Graphs with specified radius and diameter, Discrete Math. 4 (1973) 71–75.
[5] P.J. Slater, Central vertices in a graph, in: F. Hoffman et al., eds., Proc. 7th S.-E. Conf. Combinatorics, Graph Theory, and Computing (Utilitas Mathematica, Winnipeg, 1977) 487–497.
[6] V.G. Vizing, The number of edges in a graph of given radius, Soviet Math. Dokl. 8 (1967) 535–536.